Finding p.d.f of a function of random vars

$U = u(Y_1, Y_2, Y_3, \ldots Y_n)$

- Method of dist. functions (Always works)

$f_U(u) = \frac{d}{du} F_U(u)$

$Y \sim f_Y(y)$

$U = g(Y)$

$F_U(u) = P(U < u) = P(g(Y) \leq u)$

E.g. $f(y) = 2y$, $0 \leq y \leq 1$

$U = 3Y - 1$

$F_U(u) = P(3Y - 1 < u)$

$y = \frac{u+1}{3} \Rightarrow 0 \leq \frac{u+1}{3} \leq 1 \Rightarrow 0 \leq u+1 \leq 3$
$\Rightarrow -1 \leq u \leq 2$

$= \int_0^{\frac{u+1}{3}} 2y \, dy$

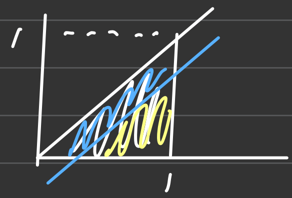
$= \left(y^2 \Big|_0^{\frac{u+1}{3}} \right) = \left(\frac{u+1}{3}\right)^2$

$f_u(u) = \frac{d}{du} F_u(u) = \frac{d}{du} \left(\frac{u+1}{3}\right)^2$

$= 2 \cdot \frac{u+1}{3} \cdot \frac{1}{3} = \frac{2u+2}{9}$ ✓ , $-1 \leq u \leq 2$ ✓

Bivariate

Example



$f(y_1, y_2) = 3y_1$ $\qquad\qquad 0 \le y_2 \le y_1 \le 1$

$U = Y_1 - Y_2$

$F_U(u) = P(Y_1 - Y_2 \le u)$

$y_2 = y_1 - u$

$0 < u < 1$

$F_U(u) = P(Y_2 \ge Y_1 - u)$

yellow

$= 1 - P(Y_2 \le Y_1 - u)$

$= 1 - \int_u^1 \int_0^{y_1 - u} 3y_1 \, dy_2 \, dy_1$

$= 1 - \int_u^1 3y_1 \times (y_1 - u) \, dy_1$

$= 1 - \int_u^1 3y_1^2 - 3uy_1 \, dy_1$

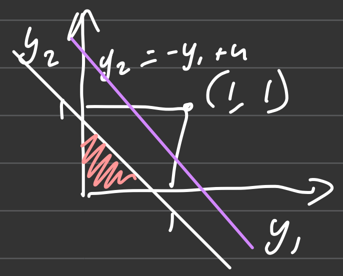
$= 1 - \left(y_1^3 - \frac{3}{2} u y_1^2 \Big|_u^1 \right)$

$= 1 - \left(1 - \frac{3}{2}u - \left(u^3 - \frac{3}{2}u^3\right)\right)$

$= 1 - \left(1 - \frac{3}{2}u + \frac{1}{2}u^3\right)$

$= \frac{3}{2}u - \frac{1}{2}u^3$

$f(y_1, y_2) = 1 \qquad 0 \le y_1, y_2 \le 1$



$U = Y_1 + Y_2$

$f(u) = \frac{d}{du} F_u(u)$

$y_2 = 1$

$y_1 = u - 1$

$F_u(u) = P(Y_1 + Y_2 \le u)$

$= P(Y_2 \le u - Y_1)$

$$= \begin{cases} \int_0^u \int_0^{-y_1+u} 1 \, dy_2 \, dy_1 & \text{for } 0 \le u < 1 \\ 1 - \int_{u-1}^{1} \int_{-y_1+u}^{1} 1 \, dy_2 \, dy_1 & \text{for } 1 \le u \le 2 \end{cases}$$

$Y \sim unif(0, 1)$

$U \sim exp(\beta)$

$U = g(Y)$

$F_u(u) = 1 - e^{-\frac{u}{\beta}}$

$y =$

Method of transformation

$Y \sim f_Y(y)$ $\qquad\qquad U = h(Y)$

$U = h(Y)$

h is monotone

$Y = h^{-1}(U)$

$$F_U(u) = P(h(Y) \le u)$$
$$= P(Y \le h^{-1}(u))$$
$$= F_Y(h^{-1}(u))$$

Know this

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} F_Y(h^{-1}(u))$$

learn proof of $| \; |$

$$= f_Y(h^{-1}(u)) \times \left| \frac{d}{du}(h^{-1}(u)) \right|$$

Method of moment generating functions

$Y \sim n(0,1) \qquad U = Y^2$

$$m_U(t) = E[e^{tU}] = E[e^{tg(Y)}]$$

$$= E[e^{tY^2}] = \int_{-\infty}^{\infty} e^{ty^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}y^2(1-2t)} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \times e^{\frac{1}{2}\left(\frac{y}{(1-2t)^{-\frac{1}{2}}}\right)^2} dy$$

$$\sigma = (1-2t)^{-\frac{1}{2}}$$

$$= (1-2t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \times (1-2t)^{-\frac{1}{2}}} \times e^{\frac{1}{2}\left(\frac{y}{(1-2t)^{-\frac{1}{2}}}\right)^2}$$

$$= (1-2t)^{-\frac{1}{2}} \times$$

If $\quad U = Y_1 + Y_2 + Y_3 + \dots$

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times m_{Y_3}(t) \times \dots$$

Proof: $\quad m_U(t) = E[e^{tU}] = E[e^{t(Y_1+Y_2+\dots)}]$

$$= E[e^{tY_1} e^{tY_2} e^{tY_3} \times \dots]$$

$$= E[e^{tY_1}] E[e^{tY_2}] E[e^{tY_3}] \times \dots$$

$E[g(x) f(y)] = E[g(x)] E[f(y)]$ if x, y independent

$Y_1, \dots, Y_n \sim exp(\Theta) \qquad E[Y_i] = \Theta$

$U = Y_1 + Y_2 + \dots Y_n$

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \dots$$

$$= e$$

WORK OUT

$Y_1, Y_2, \dots, Y_n$ independent

$$Y_i \sim n(\mu_i, \sigma_i^2)$$

$$U = \sum_i a_i Y_i$$

We know $E[U] = \sum_i a_i \mu_i$ $\quad Var(U) = \sum_i a_i^2 \sigma_i^2 + 2 \times \sum_{i,j} a_i a_j \overbrace{cov(Y_i, Y_j)}^{\text{Independent} \therefore 0}$

$$m_U(t) = \prod_i m_{Y_i}(a_i t)$$

Because $U = sY$ $\quad m_U(t) = E[e^{tsY}]$

$$= \prod_i e^{a_i \mu_i t + \frac{1}{2} t^2 \sigma_i^2 a_i^2}$$

$$= e^{\sum_i \mu_i t a_i + \frac{1}{2} t^2 \sigma_i^2 a_i^2}$$

$$= e^{t \sum \mu_i a_i + \frac{1}{2} t^2 \sum_i \sigma_i^2 a_i^2}$$

$$\therefore U \sim n\left(\sum_i \mu_i a_i, \sum_i \sigma_i^2 a_i^2\right)$$

$$Y_i \sim n(\mu_i, \sigma_i^2)$$

$$U = \sum_i \underbrace{\left(\frac{Y_i - \mu_i}{\sigma_i}\right)^2}_{V_i}$$

$$V_i \sim \chi_1^2$$

$$U = \sum_i V$$

$$m_U(t) = \prod m_{V_i}(t) = \prod_i (1-2t)^{-\frac{1}{2}}$$

$$= (1-2t)^{-\frac{n}{2}}$$

$$U \sim \chi_n^2$$

Recap

Method of dist fns

- find $F_u(u)$
- $f_u(u) = \frac{d}{du} F_u(u)$

$$F_u(u) = \int \cdots \int\int f(y_1, y_2, y_3, \dots)\, dy_1 \dots dy_n$$

Method of transformation

$$f(u) = f_Y[g^{-1}(u)] \left| \frac{d}{du} g^{-1}(u) \right|$$